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Symmetry groups and the pseudo-Riemann spacetimes for mixed-hardening elastoplasticity

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Abstract

The constitutive postulations for mixed-hardening elastoplasticity are selected. Several homeomorphisms of irreversibility parameters are derived, among which $X_{\rm a}^0$ and $X_{\rm c}^0$ play respectively the roles of temporal components of the Minkowski and conformal spacetimes. An augmented vector $\mathbf{X}_{\rm a} := (Y\mathbf{Q}_{\rm a}^{\rm t}, YQ_{\rm a}^0)^{\rm t}$ is constructed, whose governing equations in the plastic phase are found to be a linear system with a suitable rescaling proper time. The underlying structure of mixed-hardening elastoplasticity is a Minkowski spacetime $\mathbb{M}^{n+1}$ on which the proper orthochronous Lorentz group $SO_{\rm o}(n,1)$ left acts. Then, constructed is a Poincaré group $ISO_{\rm o}(n,1)$ on space $\mathbf{X} := \mathbf{X}_{\rm a} + \mathbf{X}_{\rm b}$, of which $\mathbf{X}_{\rm b}$ reflects the kinematic hardening rule in the model. We also find that the space $(\mathbf{Q}_{\rm a}^{\rm t}, q_0^a)$ is a Robertson–Walker spacetime, which is conformal to $\mathbf{X}_{\rm a}$ through a factor Y, and conformal to $\mathbf{X}_{\rm c} := (\rho\mathbf{Q}_{\rm a}^{\rm t}, \rho Q_{\rm a}^0)^{\rm t}$ through a factor ρ as given by $\rho(q_0^a) = Y(q_0^a)/[1 - 2\rho_0 Q_{\rm a}^0(0) + 2\rho_0 Y(q_0^a) Q_{\rm a}^0(q_0^a)]$. In the conformal spacetime the internal symmetry is a conformal group.

Keywords: Elastoplasticity; Mixed-hardening; Symmetry group; Minkowski spacetime; Conformal spacetime

1. Introduction

The formulation of yield criteria of Tresca and of von Mises was the first key step in developing plasticity theory. However, without a plastic stress-strain relation, the criteria can do little. The incremental stress-strain relations proposed by Saint-Venant and Lévy represented a giant step forward in plasticity theory. Since then the incremental or rate type point-of-view has been most frequently taken in formulating constitutive equations of plasticity, which describe the evolutions of internal state variables. This kind of rate-equation representations is now composed of separately specified but interwoven ingredients, including the elastic part, yield condition, loading-unloading criterion, plastic flow rule, isotropic hardening rule and kinematic hardening rule, etc.

Because of the nonlinear nature of the model ingredients in plasticity, there seems necessary although difficult by composing all ingredients together to derive a global theory of plasticity, which not only gives

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directly a local response of the material in the time domain once an input is prescribed, but also facilitates us to handle model property from a global view. To do this several thresholds need to be stridden across. The first is integrating the rate-type equations, which consist of separated but interwoven ingredients specified on a high-dimensional space such as states in stress space and paths in plastic strain space and their product space. The second is solving the irreversibility parameters, such as the equivalent plastic strain, dissipation, and so on, and in general there exist monotonic mappings between these parameters. The third as to be strongly emphasized here is studying symmetry properties of the model and its intrinsic spacetime structures.

The differential nature of plasticity laws has been discussed for a long time. The invariant yield condition in stress space renders a natural mathematical frame of plasticity theory from the view of differentiable manifold and its Lie group transformation. In this paper we are going to make an effort to study the internal symmetry groups and the underlying spacetime structures for mixed-hardening elastoplasticity. The Lie group theory provides a universal tool for tackling considerable numbers of differential equations when other means fail. Indeed, group analyses may augment intuition in understanding and in using symmetry for formulation of physical models, and often disclose possible approaches to solving complex problems.

Recently, Hong and Liu (1999a,b, 2000) have considered constitutive models of perfect elastoplasticity with and without considering large deformation and of bilinear elastoplasticity, revealing that the models possess two kinds of internal symmetries, characterized by the translation group $T(n)$ in the off (or elastic) phase and by the projective proper orthochronous Lorentz group $\mathrm{PSO}_{\mathrm{o}}(n,1)$ in the on (or elastoplastic) phase for the first two models and the proper orthochronous Poincaré group $\mathrm{PISO}_{\mathrm{o}}(n,1)$ in the on phase for the bilinear model, and have symmetry switching between the two groups dictated by the control input. Moreover, Liu (2001, 2002) applied a $\mathrm{PDSO}_{\mathrm{o}}(n,1)$ symmetry to formulate a mathematical model of visco-elastoplasticity, and then Liu (submitted for publication) employed a two-component spinor representation to unify finite strain elastic-perfectly plastic models with different objective stress rates and showed that the Lie symmetry in the two-dimensional spinor space is $SL(2,\mathbb{H})$ with $\mathbb{H}$ denoting the quaternionic number. In this paper some of those results are extended to the model of mixed-hardening elastoplasticity, investigating the influence of nonlinear hardening parameters on the structure of the underlying vector spaces and on the transformation groups of internal symmetries.

2. Constitutive postulations

In terms of five-dimensional stress vectors and five-dimensional strain vectors:

$$\mathbf{Q} = \begin{bmatrix} a_1 s^{11} + a_2 s^{22} \\ a_3 s^{11} + a_4 s^{22} \\ s^{23} \\ s^{13} \\ s^{12} \end{bmatrix}, \quad \mathbf{Q}_{\mathrm{a}} = \begin{bmatrix} a_1 s^{11}_{\mathrm{a}} + a_2 s^{22}_{\mathrm{a}} \\ a_3 s^{11}_{\mathrm{a}} + a_4 s^{22}_{\mathrm{a}} \\ s^{23}_{\mathrm{a}} \\ s^{13}_{\mathrm{a}} \\ s^{12}_{\mathrm{a}} \end{bmatrix}, \quad \mathbf{Q}_{\mathrm{b}} = \begin{bmatrix} a_1 s^{11}_{\mathrm{b}} + a_2 s^{22}_{\mathrm{b}} \\ a_3 s^{11}_{\mathrm{b}} + a_4 s^{22}_{\mathrm{b}} \\ s^{23}_{\mathrm{b}} \\ s^{13}_{\mathrm{b}} \\ s^{12}_{\mathrm{b}} \end{bmatrix}, \tag{1}$$

$$\mathbf{q} = \begin{bmatrix} a_1 e_{11} + a_2 e_{22} \\ a_3 e_{11} + a_4 e_{22} \\ e_{23} \\ e_{13} \\ e_{12} \end{bmatrix}, \quad \mathbf{q}^{\mathrm{e}} = \begin{bmatrix} a_1 e^{\mathrm{e}}_{11} + a_2 e^{\mathrm{e}}_{22} \\ a_3 e^{\mathrm{e}}_{11} + a_4 e^{\mathrm{e}}_{22} \\ e^{\mathrm{e}}_{23} \\ e^{\mathrm{e}}_{13} \\ e^{\mathrm{e}}_{12} \end{bmatrix}, \quad \mathbf{q}^{\mathrm{p}} = \begin{bmatrix} a_1 e^{\mathrm{p}}_{11} + a_2 e^{\mathrm{p}}_{22} \\ a_3 e^{\mathrm{p}}_{11} + a_4 e^{\mathrm{p}}_{22} \\ e^{\mathrm{p}}_{23} \\ e^{\mathrm{p}}_{13} \\ e^{\mathrm{p}}_{12} \end{bmatrix}, \tag{2}$$

the tensorial form of elastoplastic model with mixed-hardening rule, e.g., Hong and Liu (1993):

$$\dot{\mathbf{e}} = \dot{\mathbf{e}}^{\mathrm{e}} + \dot{\mathbf{e}}^{\mathrm{p}}, \tag{3}$$

$$\mathbf{s} = \mathbf{s}_\mathrm{a} + \mathbf{s}_\mathrm{b}, \tag{4}$$

$$\dot{\mathbf{s}} = 2G\dot{\mathbf{e}}^\mathrm{e}, \tag{5}$$

$$\dot{\lambda}\mathbf{s}_\mathrm{a} = 2\tau_y^0\dot{\mathbf{e}}^\mathrm{p}, \tag{6}$$

$$\dot{\mathbf{s}}_\mathrm{b} = 2k'\dot{\mathbf{e}}^\mathrm{p}, \tag{7}$$

$$\|\mathbf{s}_\mathrm{a}\| \leqslant \sqrt{2}\tau_y^0, \tag{8}$$

$$\dot{\lambda} \geqslant 0, \tag{9}$$

$$\|\mathbf{s}_\mathrm{a}\|\dot{\lambda} = \sqrt{2}\tau_y^0\dot{\lambda}, \tag{10}$$

can be equivalently re-expressed by the following vectorial form:

$$\dot{\mathbf{q}} = \dot{\mathbf{q}}^\mathrm{e} + \dot{\mathbf{q}}^\mathrm{p}, \tag{11}$$

$$\mathbf{Q} = \mathbf{Q}_\mathrm{a} + \mathbf{Q}_\mathrm{b}, \tag{12}$$

$$\dot{\mathbf{Q}} = k_\mathrm{e}\dot{\mathbf{q}}^\mathrm{e}, \tag{13}$$

$$\mathbf{Q}_\mathrm{a}\dot{q}_0^a = Q_\mathrm{a}^0\dot{\mathbf{q}}^\mathrm{p}, \tag{14}$$

$$\dot{\mathbf{Q}}_\mathrm{b} = k_\mathrm{p}\dot{\mathbf{q}}^\mathrm{p}, \tag{15}$$

$$\|\mathbf{Q}_\mathrm{a}\| \leqslant Q_\mathrm{a}^0, \tag{16}$$

$$\dot{q}_0^a \geqslant 0, \tag{17}$$

$$\|\mathbf{Q}_\mathrm{a}\|\dot{q}_0^a = Q_\mathrm{a}^0\dot{q}_0^a. \tag{18}$$

In Eqs. (1) and (2),

$$a_1 := \sin\left(\theta + \frac{\pi}{3}\right), \quad a_2 := \sin\theta, \quad a_3 := \cos\left(\theta + \frac{\pi}{3}\right), \quad a_4 := \cos\theta, \tag{19}$$

where θ can be any real number. If choosing $\theta = 0$ we have the stress space $\mathbf{Q} := (\sqrt{3}s^{11}/2, s^{11}/2 + s^{22}, s^{23}, s^{13}, s^{12})^\mathrm{t}$ of Il'yushin (1963). Throughout this paper, a superscript t indicates the transpose, and a superimposed dot denotes a differentiation with respect to time.

In the tensorial representation of elastoplastic model with Prager's kinematic hardening rule combined with isotropic hardening rule as shown by Eqs. (3)–(10), $\mathbf{e}$, $\mathbf{e}^\mathrm{e}$, $\mathbf{e}^\mathrm{p}$, $\mathbf{s}$, $\mathbf{s}_\mathrm{a}$ and $\mathbf{s}_\mathrm{b}$ are, respectively, the deviatoric tensors of strain, elastic strain, plastic strain, stress, active stress and back stress, all symmetric and traceless, whereas λ is a scalar evaluated by

$$\lambda(t) = \int_0^t \sqrt{2}\|\dot{\mathbf{e}}^\mathrm{p}(\xi)\|\,\mathrm{d}\xi, \tag{20}$$

where $\|\dot{\mathbf{e}}^\mathrm{p}\| := \sqrt{\dot{\mathbf{e}}^\mathrm{p} \cdot \dot{\mathbf{e}}^\mathrm{p}}$ defines the Euclidean norm of $\dot{\mathbf{e}}^\mathrm{p}$ and a dot between two tensors stands for their inner product. From Eqs. (1) and (2) it follows that

$$\|\mathbf{s}_\mathrm{a}\|^2 = 2\|\mathbf{Q}_\mathrm{a}\|^2, \quad \mathbf{s}_\mathrm{a} \cdot \dot{\mathbf{e}} = 2\mathbf{Q}_\mathrm{a}^\mathrm{t}\dot{\mathbf{q}}. \tag{21}$$

The material functions and plastic multipliers in the above two equivalent representations have the following relations:

$$k_e = 2G, \quad k_p = 2k', \quad Q_a^0 = \tau_y^0, \quad 2\dot{q}_0^a = \dot{\lambda}. \tag{22}$$

The norm of an n-vector $\mathbf{Q}$ is defined as $\|\mathbf{Q}\| := \sqrt{\mathbf{Q}^t\mathbf{Q}}$. Depending on the number of nonzero stress components in Eq. (1) (and correspondingly nonzero strain components in Eq. (2)) which we consider for a physical problem, for example, the axial tension-compression problem, the biaxial tension-compression-torsion problem, etc., the dimension n may be an integer with $1 \leqslant n \leqslant 5$, and no matter which case is we use n to denote the physical problem dimension. The bold-faced letters $\mathbf{q}$, $\mathbf{q}^e$, $\mathbf{q}^p$, $\mathbf{Q}$, $\mathbf{Q}_a$ and $\mathbf{Q}_b$ are, respectively, the n-vectors of generalized strain, generalized elastic strain, generalized plastic strain, generalized stress, generalized active stress and generalized back stress, whereas q_0^a is a scalar. All $\mathbf{q}$, $\mathbf{q}^e$, $\mathbf{q}^p$, $\mathbf{Q}_a$, $\mathbf{Q}_b$, $\mathbf{Q}$ and q_0^a are functions of one and the same independent variable, which in most cases is taken either as the usual time or as the arc length of the controlled generalized strain path; however, for convenience, the independent variable no matter what it is will be simply called "time" and given the notation t. The generalized elastic modulus $k_e > 0$, generalized kinematic modulus k_p, and the generalized yield strength $Q_a^0 > 0$ are the only three property functions needed in the model, and are functions of the equivalent generalized plastic strain q_0^a, which by Eqs. (14), (17) and (18) and $Q_a^0 > 0$ is found to be

$$q_0^a = \int_0^t \dot{q}_0^a(\xi)\,d\xi = \int_0^t \|\dot{\mathbf{q}}^p(\xi)\|\,d\xi. \tag{23}$$

The material is further assumed to be weakly stable (Hong and Liu, 1993):

$$k_e + k_p + Q_a^{0'} > 0, \quad k_e + k_p > 0. \tag{24}$$

Here a prime denotes the derivative of a function with respect to its argument, for example, $Q_a^{0'} := dQ_a^0(q_0^a)/dq_0^a$.

3. Response operators

From Eqs. (11)–(15) it follows that

$$\dot{\mathbf{Q}}_a + \frac{k_e + k_p}{Q_a^0}\dot{q}_0^a\mathbf{Q}_a = k_e\dot{\mathbf{q}}, \tag{25}$$

which, in terms of the integrating factor

$$Y(q_0^a) := \exp\left[\int_0^{q_0^a} \frac{k_e(p) + k_p(p)}{Q_a^0(p)}\,dp\right], \tag{26}$$

can be integrated to

$$\mathbf{Q}_a(t) = \frac{1}{Y(q_0^a(t))}\left[Y(q_0^a(t_i))\mathbf{Q}_a(t_i) + \int_{t_i}^t k_e(q_0^a(\xi))Y(q_0^a(\xi))\dot{\mathbf{q}}(\xi)\,d\xi\right]. \tag{27}$$

Upon substituting Eq. (27) for $\mathbf{Q}_a$ into Eq. (14) and integrating the resultant, we obtain

$$\mathbf{q}^p(t) = \mathbf{q}^p(t_i) + \frac{G^p(q_0^a(t), q_0^a(t_i))}{k_e(q_0^a(t_i))}\mathbf{Q}_a(t_i) + \int_{t_i}^t G^p(q_0^a(t), q_0^a(\xi))\dot{\mathbf{q}}(\xi)\,d\xi, \tag{28}$$

where

$$C(q_0^a) := \int_0^{q_0^a} \frac{1}{Y(p)Q_a^0(p)} \, dp, \tag{29}$$

$$G^p(p_1, p_2) := k_e(p_2)Y(p_2)[C(p_1) - C(p_2)]. \tag{30}$$

Integrating Eq. (13) after replaced its $\dot{\mathbf{q}}^e$ by $\dot{\mathbf{q}} - \dot{\mathbf{q}}^p$ due to Eq. (11), $\dot{\mathbf{q}}^p$ by the flow rule (14) and $\mathbf{Q}_a$ by Eq. (27), we obtain

$$\mathbf{Q}(t) = \mathbf{Q}(t_i) + \left[G^s(q_0^a(t), q_0^a(t_i)) - G^s(q_0^a(t_i), q_0^a(t_i))\right] \frac{\mathbf{Q}_a(t_i)}{k_e(q_0^a(t_i))} + \int_{t_i}^{t} G^s(q_0^a(t), q_0^a(\xi))\dot{\mathbf{q}}(\xi) \, d\xi, \tag{31}$$

where

$$C_e(q_0^a) := \int_0^{q_0^a} \frac{k_e(p)}{Y(p)Q_a^0(p)} \, dp, \tag{32}$$

$$G^s(p_1, p_2) := k_e(p_2)\{1 - Y(p_2)[C_e(p_1) - C_e(p_2)]\}. \tag{33}$$

Up to now three integral operators (27), (28) and (31) are established which map respectively generalized strain rate histories into generalized active stress histories, generalized plastic strain histories and generalized stress histories. Here t is the (current) time and t_i is an initial time, at which initial conditions $\mathbf{Q}_a(t_i)$, $\mathbf{q}^p(t_i)$, $\mathbf{Q}(t_i)$ and $q_0^a(t_i)$ should be prescribed. Obviously, the expressions in Eqs. (27), (28) and (31) that the responses are in terms of the generalized strain rate history is useful only if the history of q_0^a is already known. Thus q_0^a deserves a more study, as will be pursued in the following two sections.

4. Switch of plastic irreversibility

The complementary trios (16)–(18) enable the model to possess a switch of plastic irreversibility, the criteria for which are derived right below.

Taking the inner product of Eq. (25) with $\mathbf{Q}_a$, we get

$$\beta \mathbf{Q}_a^t \dot{\mathbf{q}} = Q_a^0 \dot{q}_0^a \quad \text{if } \|\mathbf{Q}_a\| = Q_a^0, \tag{34}$$

where

$$\beta := \frac{k_e}{k_e + k_p + Q_a^{0\prime}} > 0 \tag{35}$$

is a material function. Since $\beta > 0$ and $Q_a^0 > 0$, Eq. (34) assures that

$$\text{if } \|\mathbf{Q}_a\| = Q_a^0 \quad \text{then} \quad \mathbf{Q}_a^t \dot{\mathbf{q}} > 0 \iff \dot{q}_0^a > 0. \tag{36}$$

Hence,

$$\{\|\mathbf{Q}_a\| = Q_a^0 \text{ and } \mathbf{Q}_a^t \dot{\mathbf{q}} > 0\} \Rightarrow \dot{q}_0^a > 0. \tag{37}$$

On the other hand, if $\dot{q}_0^a > 0$, Eq. (18) assures $\|\mathbf{Q}_a\| = Q_a^0$, which together with Eq. (36) asserts that

$$\dot{q}_0^a > 0 \Rightarrow \{\|\mathbf{Q}_a\| = Q_a^0 \text{ and } \mathbf{Q}_a^t \dot{\mathbf{q}} > 0\}. \tag{38}$$

Therefore, from Eqs. (37) and (38) we conclude that the yield condition $\|\mathbf{Q}_a\| = Q_a^0$ and the straining condition $\mathbf{Q}_a^t \dot{\mathbf{q}} > 0$ are sufficient and necessary for plastic irreversibility $\dot{q}_0^a > 0$.

Considering this and the inequality (17), we thus reveal the following criteria of plastic irreversibility:

$$\dot{q}_0^a = \begin{cases} \frac{\beta}{Q_a^0} \mathbf{Q}_a^t \dot{\mathbf{q}} > 0 & \text{if } \|\mathbf{Q}_a\| = Q_a^0 \text{ and } \mathbf{Q}_a^t \dot{\mathbf{q}} > 0, \quad \boxed{\text{ON}} \\ 0 & \text{if } \|\mathbf{Q}_a\| < Q_a^0 \text{ or } \mathbf{Q}_a^t \dot{\mathbf{q}} \leqslant 0. \quad \boxed{\text{OFF}} \end{cases} \tag{39}$$

In the $\boxed{\text{ON}}$ phase of the switch, $\dot{q}_0^a > 0$, the mechanism of plastic irreversibility is working and the material exhibits elastoplastic behavior, while in the $\boxed{\text{OFF}}$ phase of the switch, $\dot{q}_0^a = 0$, the material behavior is reversible and elastic. According to the complementary trios (16)–(18), there are just two phases: ① $\dot{q}_0^a > 0$ and $\|\mathbf{Q}_a\| = Q_a^0$, and ② $\dot{q}_0^a = 0$ and $\|\mathbf{Q}_a\| \leqslant Q_a^0$. From the switch (39) it is clear that ① corresponds to the $\boxed{\text{ON}}$ phase whereas ② to the $\boxed{\text{OFF}}$ phase.

5. Measure of plastic irreversibility

The switch (39) together with the relation (26) reveals the core importance of q_0^a and Y in the whole model; we now go further to investigate their evolutions. Substitution of Eq. (27) into Eq. (34) and rearrangement yield

$$\frac{1}{\beta} Y Q_a^0 \dot{q}_0^a = Y(q_0^a(t_i)) \mathbf{Q}_a^t(t_i) \dot{\mathbf{q}}(t) + \int_{t_i}^t k_e(q_0^a(\xi)) Y(q_0^a(\xi)) \dot{\mathbf{q}}^t(t) \dot{\mathbf{q}}(\xi) \, d\xi. \tag{40}$$

Let

$$Z(q_0^a) := \int_0^{q_0^a} \frac{Y(p) Q_a^0(p)}{\beta(p)} \, dp, \tag{41}$$

such that

$$\dot{Z} = Z' \dot{q}_0^a = \frac{Y Q_a^0}{\beta} \dot{q}_0^a. \tag{42}$$

From the inequality (17) and $Q_a^0 > 0$, $\beta > 0$, and $Y > 0$ in view of the definition (26), we have

$$\dot{Z} \geqslant 0, \quad Z' = \frac{Y Q_a^0}{\beta} > 0, \tag{43}$$

which ensures the invertibility of the material function $Z(q_0^a)$. Hence,

$$q_0^a = q_0^a(Z) \tag{44}$$

is available, and all the material functions that originally expressed in terms of q_0^a can be re-expressed in terms of Z. However, for saving notations we use the same symbols to denote such new functions, for example,

$$Y(Z) := Y(q_0^a(Z)). \tag{45}$$

Substituting Eqs. (42) and (44) into Eq. (40), we obtain

$$\dot{Z}(t) = Y(Z(t_i)) \mathbf{Q}_a^t(t_i) \dot{\mathbf{q}}(t) + \int_{t_i}^t k_e(Z(\xi)) Y(Z(\xi)) \dot{\mathbf{q}}^t(t) \dot{\mathbf{q}}(\xi) \, d\xi. \tag{46}$$

Then, integration gives

$$Z(t) = Z(t_i) + Y(Z(t_i)) \mathbf{Q}_a^t(t_i) [\mathbf{q}(t) - \mathbf{q}(t_i)] + \int_{t_i}^t k_e(Z(\xi)) Y(Z(\xi)) [\mathbf{q}(t) - \mathbf{q}(\xi)]^t \dot{\mathbf{q}}(\xi) \, d\xi, \tag{47}$$

which is a nonlinear Volterra integral equation for Z.

6. Homeomorphisms of measures of plastic irreversibility

From Eqs. (26), (42), $(43)_1$ and $(24)_2$ it follows that

$$\dot{Y} = \frac{\beta(k_e + k_p)}{(Q_a^0)^2}\dot{Z} \geqslant 0, \tag{48}$$

and the dissipation power is

$$\dot{\Lambda} := \mathbf{Q}_a^t \dot{\mathbf{q}}^p = Q_a^0 \dot{q}_0^a \geqslant 0. \tag{49}$$

In view of the switch (39), inequalities (17) and $(43)_1$, and Eqs. (42), (48) and (49), it is important to note that q_0^a, Y, Z and Λ are intimately related in the sense that there exists strictly monotonic increasing relation between any pair of them as shown in Table 1. Therefore, any one of those irreversibility parameters can be chosen to play the role in the switch (39), the role of an indicator of irreversible change of material properties, the role of so-called material age, intrinsic time, internal time, the arrow of time, etc., in this time-independent model. Furthermore, they serve as the measures of plastic irreversibility and are crucially important in the evolutions of material properties and parameters and also in the determination of responses as shown in Eqs. (27), (28) and (31). Once Z is obtained, the other parameters are readily calculable through the integrals of the material homeomorphic functions listed in Table 1. Hence, Eq. (47) or Eq. (46) deserves a further study.

7. Material functions in terms of Z

By Eqs. (41) and (44) the irreversibility parameter Z and the equivalent generalized plastic strain q_0^a are closely related, being bijective, invertible, and strictly monotonic. This fact of equivalence can be utilized to accelerate the calculation of the material responses. For the generalized strain-controlled processes we may thereby change the dependence of the defined material functions on q_0^a to directly on Z, and thus all the material functions with the arguments q_0^a turn to the material functions simply with the arguments Z through Eq. (44). Even the process listed in the above is available to reveal the function dependence of $Q_a^0(Z)$, $k_p(Z)$ and so on. But they are not so straightforward. In this section the relationships of the material functions are further studied from a different point-of-view.

Table 1
Relationships between irreversibility parameters

	$\dot{q}_0^a$	$\dot{Y}$	$\dot{Z}$	$\dot{\Lambda}$	$\dot{X}_a^0$	$\dot{X}_c^0$
$\dot{q}_0^a$	1	$\frac{Q_a^0}{(k_e+k_p)Y}$	$\frac{\beta}{Q_a^0 Y}$	$\frac{1}{Q_a^0}$	$\frac{\beta}{k_e Y}$	$\frac{\beta[1-2\rho_0 Q_a^0(0)]}{k_e Y[1-2\rho_0 X_c^0]^2}$
$\dot{Y}$	$\frac{(k_e+k_p)Y}{Q_a^0}$	1	$\frac{\beta(k_e+k_p)}{(Q_a^0)^2}$	$\frac{(k_e+k_p)Y}{(Q_a^0)^2}$	$\frac{\beta(k_e+k_p)}{k_e Q_a^0}$	$\frac{\beta(k_e+k_p)[1-2\rho_0 Q_a^0(0)]}{k_e Q_a^0[1-2\rho_0 X_c^0]^2}$
$\dot{Z}$	$\frac{Q_a^0 Y}{\beta}$	$\frac{(Q_a^0)^2}{\beta(k_e+k_p)}$	1	$\frac{Y}{\beta}$	$\frac{Q_a^0}{k_e}$	$\frac{Q_a^0[1-2\rho_0 Q_a^0(0)]}{k_e[1-2\rho_0 X_c^0]^2}$
$\dot{\Lambda}$	Q_a^0	$\frac{(Q_a^0)^2}{(k_e+k_p)Y}$	$\frac{\beta}{Y}$	1	$\frac{\beta Q_a^0}{k_e Y}$	$\frac{\beta Q_a^0[1-2\rho_0 Q_a^0(0)]}{k_e Y[1-2\rho_0 X_c^0]^2}$
$\dot{X}_a^0$	$\frac{k_e Y}{\beta}$	$\frac{k_e Q_a^0}{\beta(k_e+k_p)}$	$\frac{k_e}{Q_a^0}$	$\frac{k_e Y}{\beta Q_a^0}$	1	$\frac{1-2\rho_0 Q_a^0(0)}{[1-2\rho_0 X_c^0]^2}$
$\dot{X}_c^0$	$\frac{k_e Y[1-2\rho_0 X_c^0]^2}{\beta[1-2\rho_0 Q_a^0(0)]}$	$\frac{k_e Q_a^0[1-2\rho_0 X_c^0]^2}{\beta(k_e+k_p)[1-2\rho_0 Q_a^0(0)]}$	$\frac{k_e[1-2\rho_0 X_c^0]^2}{Q_a^0[1-2\rho_0 Q_a^0(0)]}$	$\frac{k_e Y[1-2\rho_0 X_c^0]^2}{\beta Q_a^0[1-2\rho_0 Q_a^0(0)]}$	$\frac{[1-2\rho_0 X_c^0]^2}{1-2\rho_0 Q_a^0(0)}$	1

In order to give the dependence of the material functions on Z directly, we combine Eqs. (45) and (26) to get

$$\exp\left[\int_0^{q_0^a}\frac{k_e(p)+k_p(p)}{Q_a^0(p)}\,dp\right]=Y(Z). \tag{50}$$

Derivating both sides with respect to q_0^a, applying the chain rule on the right-hand side and then using Eq. $(43)_2$, we obtain

$$\frac{\beta(q_0^a)[k_e(q_0^a)+k_p(q_0^a)]}{(Q_a^0(q_0^a))^2}=Y'(Z). \tag{51}$$

Similarly, the derivative of Eq. (32) with respect to q_0^a and the use of Eq. $(43)_2$ give

$$\frac{\beta(q_0^a)k_e(q_0^a)}{(Q_a^0(q_0^a))^2}=Y^2(Z)C_e'(Z). \tag{52}$$

By dividing the above two equations we obtain

$$k_e(q_0^a)+k_p(q_0^a)=\frac{k_e(Z)Y'(Z)}{Y^2(Z)C_e'(Z)}, \tag{53}$$

or in terms of Z,

$$k_p(Z)=k_e(Z)\left[\frac{Y'(Z)}{Y^2(Z)C_e'(Z)}-1\right]. \tag{54}$$

The replacement of β in Eq. (52) by the one in Eq. (35) and the substitution of Eq. (53) into the resultant yield

$$\frac{k_e^2(Z)}{Y^2(Z)C_e'(Z)}=(Q_a^0(q_0^a))^2\left[Q_a^{0'}(q_0^a)+\frac{k_e(Z)Y'(Z)}{Y^2(Z)C_e'(Z)}\right]. \tag{55}$$

By the chain rule and the use of Eq. $(43)_2$ the term $Q_a^{0'}(q_0^a)$ in the above equation can be written as

$$Q_a^{0'}(q_0^a)=Q_a^{0'}(Z)\frac{Y(Z)Q_a^0(Z)}{\beta(Z)}. \tag{56}$$

Upon using Eqs. (53) and (35) it can be arranged to

$$Q_a^{0'}(q_0^a)=\frac{Q_a^0(Z)Q_a^{0'}(Z)\frac{k_e(Z)Y'(Z)}{Y(Z)C_e'(Z)}}{k_e(Z)-Y(Z)Q_a^0(Z)Q_a^{0'}(Z)}, \tag{57}$$

which is then substituted into Eq. (55), giving

$$\frac{k_e^2(Z)}{Y^2(Z)C_e'(Z)}=(Q_a^0(Z))^2\left[\frac{Q_a^0(Z)Q_a^{0'}(Z)\frac{k_e(Z)Y'(Z)}{Y(Z)C_e'(Z)}}{k_e(Z)-Y(Z)Q_a^0(Z)Q_a^{0'}(Z)}+\frac{k_e(Z)Y'(Z)}{Y^2(Z)C_e'(Z)}\right]. \tag{58}$$

With some algebraic manipulations the following differential equation relating $Y(Z)$ and $(Q_a^0(Z))^2$ can be obtained,

$$\frac{d}{dZ}(Q_a^0(Z))^2+\frac{2Y'(Z)}{Y(Z)}(Q_a^0(Z))^2=\frac{k_e(Z)}{Y(Z)}. \tag{59}$$

This equation provides an interesting relation among the three material functions $k_e(Z)$, $Q_a^0(Z)$ and $Y(Z)$.

If $Y(Z)$ and $k_e(Z)$ are given then the solution of $(Q_a^0(Z))^2$ obtained from Eq. (59) is

$$(Q_a^0(Z))^2 = \frac{2}{Y^2(Z)} \int_0^Z k_e(p) Y(p)\,\mathrm{d}p + \frac{(Q_a^0(0))^2}{Y^2(Z)}, \tag{60}$$

and, if $Q_a^0(Z)$ and $k_e(Z)$ are given then the solution of $Y(Z)$ obtained from Eq. (59) is

$$Y(Z) = \frac{1}{Q_a^0(Z)} \int_0^Z \frac{k_e(p)}{Q_a^0(p)}\,\mathrm{d}p + \frac{Q_a^0(0)}{Q_a^0(Z)}. \tag{61}$$

$C_e(Z)$ is calculated from Eq. (54) by

$$C_e(Z) = \int_0^Z \frac{k_e(p) Y'(p)}{[k_e(p) + k_p(p)] Y^2(p)}\,\mathrm{d}p, \tag{62}$$

and $k_p(Z)$ is evaluated by Eq. (54). Similarly,

$$C(Z) := \int_0^Z \frac{C_e'(p)}{k_e(p)}\,\mathrm{d}p \tag{63}$$

is obtained from Eq. (29) by changing the integration variable by means of Eqs. (42) and (52).

Eq. (60) provides a functional relation between $(YQ_a^0)^2$ and $k_e Y$, and Eq. (61) a functional relation between YQ_a^0 and k_e/Q_a^0. The above two equations are consistent. Interestingly, the parameter YQ_a^0 has already appeared in several places, for example, Eqs. (29), (32), (41)–(43), etc. In Section 9 we will reveal the importance of the parameter YQ_a^0 for further understanding of the mixed-hardening plasticity model.

8. Operators in terms of Z

Once Z is obtained, the material functions $Q_a^0(Z)$ and $k_p(Z)$ are specified, and $Y(Z)$ and $C_e(Z)$ are evaluated via Eqs. (61) and (62) (or $Y(Z)$ and $C_e(Z)$ are specified, and $Q_a^0(Z)$ and $k_p(Z)$ are evaluated via Eqs. (60) and (54)), $\mathbf{Q}_a(t)$ and $\mathbf{Q}(t)$ can be calculated respectively by

$$\mathbf{Q}_a(t) = \frac{1}{Y(Z(t))} \left[Y(Z(t_i)) \mathbf{Q}_a(t_i) + \int_{t_i}^{t} k_e(Z(\xi)) Y(Z(\xi)) \dot{\mathbf{q}}(\xi)\,\mathrm{d}\xi \right], \tag{64}$$

$$\mathbf{Q}(t) = \mathbf{Q}(t_i) + [G^s(Z(t), Z(t_i)) - G^s(Z(t_i), Z(t_i))] \frac{\mathbf{Q}_a(t_i)}{k_e(Z(t_i))} + \int_{t_i}^{t} G^s(Z(t), Z(\xi)) \dot{\mathbf{q}}(\xi)\,\mathrm{d}\xi, \tag{65}$$

where

$$G^s(Z_1, Z_2) := k_e(Z_2)\{1 - Y(Z_2)[C_e(Z_1) - C_e(Z_2)]\}. \tag{66}$$

Similarly, the generalized plastic strain is calculated by

$$\mathbf{q}^p(t) = \mathbf{q}^p(t_i) + \frac{G^p(Z(t), Z(t_i))}{k_e(Z(t_i))} \mathbf{Q}_a(t_i) + \int_{t_i}^{t} G^p(Z(t), Z(\xi)) \dot{\mathbf{q}}(\xi)\,\mathrm{d}\xi, \tag{67}$$

where

$$G^p(Z_1, Z_2) := k_e(Z_2) Y(Z_2)[C(Z_1) - C(Z_2)]. \tag{68}$$

A numerical procedure based on the discretizations of Eqs. (47), (64), (65) and (67) can be developed to calculate the responses of mixed-hardening elastoplastic model under general loading conditions. However, this important computational issue is reported by Liu (submitted for publication) in other place.

9. Augmented differential equations system

From Eqs. (46) and (64) we have [1]

$$\dot{Z} = Y(Z)\mathbf{Q}_a^t\dot{\mathbf{q}}, \tag{69}$$

and from Eqs. (25), (26), (44) and (45) we have

$$\frac{d}{dt}[Y(Z)\mathbf{Q}_a] = k_e(Z)Y(Z)\dot{\mathbf{q}}, \tag{70}$$

which can be rearranged to

$$\frac{d}{dt}[Y(Z)\mathbf{Q}_a] = Y(Z)Q_a^0(Z)\frac{k_e(Z)}{Q_a^0(Z)}\dot{\mathbf{q}}. \tag{71}$$

On the other hand, multiplying Eq. (61) by Q_a^0 on both sides, taking the time derivative and then using Eq. (69) we obtain

$$\frac{d}{dt}[Y(Z)Q_a^0(Z)] = Y(Z)\frac{k_e(Z)}{Q_a^0(Z)}\mathbf{Q}_a^t\dot{\mathbf{q}}. \tag{72}$$

Let us introduce

$$\mathbf{X}_a = \begin{bmatrix} \mathbf{X}_a^s \\ X_a^0 \end{bmatrix} := \begin{bmatrix} Y\mathbf{Q}_a \\ YQ_a^0 \end{bmatrix} \tag{73}$$

as the augmented $(n+1)$-dimensional state vector, [2] where X_a^0 is another irreversible parameter, and Table 1 shows its relations with the other irreversible parameters. Correspondingly, $\mathbf{X}_a^s := Y\mathbf{Q}_a$ is the spatial part of $\mathbf{X}_a$ in the Minkowski spacetime, and we may call it the augmented generalized active stress. Moreover, from Eqs. (39), (72) and (73) it follows that

$$\dot{X}_a^0 = \begin{cases} \frac{k_e X_a^0}{(Q_a^0)^2}\mathbf{Q}_a^t\dot{\mathbf{q}} > 0 & \text{if } \|\mathbf{Q}_a\| = Q_a^0 \text{ and } \mathbf{Q}_a^t\dot{\mathbf{q}} > 0, \quad \boxed{\text{ON}} \\ 0 & \text{if } \|\mathbf{Q}_a\| < Q_a^0 \text{ or } \mathbf{Q}_a^t\dot{\mathbf{q}} \leqslant 0. \quad \boxed{\text{OFF}} \end{cases} \tag{74}$$

Thus, Eqs. (71) and (72) can be combined together as following equations system:

$$\dot{\mathbf{X}}_a = \mathbf{A}\mathbf{X}_a, \tag{75}$$

where

$$\mathbf{A} := \frac{k_e}{Q_a^0}\begin{bmatrix} \mathbf{0}_{n\times n} & \dot{\mathbf{q}} \\ \dot{\mathbf{q}}^t & 0 \end{bmatrix}. \tag{76}$$

Define the proper time τ as follows:

$$d\tau = \frac{k_e}{Q_a^0}dt, \quad \tau = \tau_i + \int_{t_i}^{t}\frac{k_e(\xi)}{Q_a^0(\xi)}d\xi, \tag{77}$$

[1] Because $Y(Z)\mathbf{Q}_a$ is the augmented generalized active stress, we may correspondingly call $\dot{Z}$ the augmented generalized active power.

[2] From Eqs. (73) and (61) we have $\dot{X}_a^0 = k_e\dot{Z}/Q_a^0 > 0$ in the on phase, and thus X_a^0 is a time-like parameter which can be viewed as the temporal component of $\mathbf{X}_a$ in the Minkowski spacetime.

where t_i is an initial time and τ_i is the corresponding initial proper time. Since the term k_e/Q_a^0 in Eq. (76) is positive, the proper time τ is a monotonic function of the external time t. The on-off switching criteria in Eq. (74) can thus be written as

$$\frac{dX_a^0}{d\tau} = \begin{cases} \frac{X_a^0}{Q_a^0}\mathbf{Q}_a^t\dot{\mathbf{q}} > 0 & \text{if } \|\mathbf{Q}_a\| = Q_a^0 \text{ and } \mathbf{Q}_a^t\dot{\mathbf{q}} > 0, \quad \boxed{\text{ON}} \\ 0 & \text{if } \|\mathbf{Q}_a\| < Q_a^0 \text{ or } \mathbf{Q}_a^t\dot{\mathbf{q}} \leqslant 0. \quad \boxed{\text{OFF}} \end{cases} \tag{78}$$

In terms of the new time scale τ, Eq. (75) becomes

$$\frac{d}{d\tau}\mathbf{X}_a = \mathbf{B}\mathbf{X}_a, \tag{79}$$

where

$$\mathbf{B} := \begin{bmatrix} \mathbf{0}_{n\times n} & \dot{\mathbf{q}} \\ \dot{\mathbf{q}}^t & 0 \end{bmatrix}. \tag{80}$$

It can be seen that in this augmented space $\mathbf{X}_a = (Y\mathbf{Q}_a^t, YQ_a^0)^t$, the governing equations become linear with respect to τ and are more tractable than the original nonlinear equations.

From Eqs. (79) and (80) a deeper understanding of the underlying structure of the model may be achieved as to be done in Sections 10–12.

10. The Minkowski spacetime

In Section 9 we found that even the constitutive equations are nonlinear in the n-dimensional state space of generalized active stresses $\mathbf{Q}_a$, but they can be transformed to linear differential equations in the $(n+1)$-dimensional augmented state space of $\mathbf{X}_a$ through a time scaling. In the augmented space not only the nonlinearity of the model is unfolded, but also an intrinsic spacetime structure of the Minkowskian type will be brought out.

Now we translate the mixed-hardening elastoplastic model postulated in Section 1 in the state space of $\mathbf{Q}_a$ to one in the augmented state space of $\mathbf{X}_a$. Accordingly the first row in Eq. (79), and Eqs. (18), (16) and (17) become

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \mathbf{X}_a^t\mathbf{g}\mathbf{X}_a \end{bmatrix} \frac{d\mathbf{X}_a}{d\tau} = \begin{bmatrix} \mathbf{0}_{n\times n} & \dot{\mathbf{q}} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} \mathbf{X}_a, \tag{81}$$

$$\mathbf{X}_a^t\mathbf{g}\mathbf{X}_a \leqslant 0, \tag{82}$$

$$\frac{dX_a^0}{d\tau} \geqslant 0, \tag{83}$$

in terms of the Minkowski metric (in the space-like convention)

$$\mathbf{g} := \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix}, \tag{84}$$

where $\mathbf{I}_n$ is the identity matrix of order n. The vector space of augmented states $\mathbf{X}_a$ endowed with the Minkowski metric tensor $\mathbf{g}$ is referred to as *Minkowski spacetime*, and designated as $\mathbb{M}^{n+1}$.

Regarding Eqs. (16) and (82), we further distinguish two correspondences:

$$\|\mathbf{Q}_a\| = Q_a^0 \Longleftrightarrow \mathbf{X}_a^t\mathbf{g}\mathbf{X}_a = 0, \tag{85}$$

$$\|\mathbf{Q}_a\| < Q_a^0 \Longleftrightarrow \mathbf{X}_a^t \mathbf{g} \mathbf{X}_a < 0. \tag{86}$$

That is, a generalized active stress state $\mathbf{Q}_a$ on (resp. within) the yield hypersphere $\|\mathbf{Q}_a\| = Q_a^0$ in the generalized active stress space of $(Q_a^1, \ldots, Q_a^n)$ corresponds to an augmented state $\mathbf{X}_a$ on the cone $\{\mathbf{X}_a | \mathbf{X}_a^t \mathbf{g} \mathbf{X}_a = 0\}$ of Minkowski spacetime (resp. in the interior $\{\mathbf{X}_a | \mathbf{X}_a^t \mathbf{g} \mathbf{X}_a < 0\}$ of the cone). The exterior $\{\mathbf{X}_a | \mathbf{X}_a^t \mathbf{g} \mathbf{X}_a > 0\}$ of the cone is uninhabitable since $\|\mathbf{Q}_a\| > Q_a^0$ is forbidden. Even though it admits an infinite number of Riemannian metrics, the yield hypersphere $\mathbb{S}^{n-1}$ of $\mathbf{Q}_a$ does not admit a Minkowskian metric, nor does the space of $(\mathbf{Q}_a^t, Y)$. It is the cone of $\mathbf{X}_a$ which admits the Minkowski metric.

From Eqs. (34), (73) and (84) it follows that

$$\|\mathbf{Q}_a\| = Q_a^0 \Rightarrow \mathbf{X}_a^t \mathbf{g} (\dot{\mathbf{q}}^t, \dot{q}_0^a/\beta)^t = 0. \tag{87}$$

Moreover, by Eqs. (73), $(78)_1$, (79) and (84) we can prove that

$$\{\|\mathbf{Q}_a\| = Q_a^0 \text{ and } \mathbf{Q}_a^t \dot{\mathbf{q}} > 0\} \Rightarrow \mathbf{X}_a^t \mathbf{g} \frac{d\mathbf{X}_a}{d\tau} = 0. \tag{88}$$

If the model is in the on phase (i.e., not only $\|\mathbf{Q}_a\| = Q_a^0$ but also $\mathbf{Q}_a^t \dot{\mathbf{q}} > 0$), then from Eqs. (85), (87) and (88) we assert that for an $\mathbf{X}_a$-path on the cone, the augmented state $\mathbf{X}_a$ is M-orthogonal to itself, its tangent vector $d\mathbf{X}_a/d\tau$ and also its dual $(\dot{\mathbf{q}}^t, \dot{q}_0^a/\beta)$. The so called M-orthogonality is an orthogonality of two $(n+1)$-vectors with respect to metric (84) in Minkowski spacetime $\mathbb{M}^{n+1}$ (see, e.g., Das, 1993 and Naber, 1992).

On the other hand, X_a^0 is frozen in the off phase as indicated by Eq. $(78)_2$ and the augmented state $\mathbf{X}_a$ stays in the closed n-disc $\mathbb{D}^n$ (i.e., closed n-ball $\mathbb{B}^n$) on the hyperplane $X_a^0 =$ constant in the space of $(X_a^1, \ldots, X_a^n, X_a^0)$, the hyperplane being identified to be Euclidean n-space $\mathbb{E}^n$, which is endowed with the Euclidean metric $\mathbf{I}_n$. In summary, the augmented state $\mathbf{X}_a$ either evolves on the cone when in the on phase or stays in the discs of simultaneity, which are stacked up one by one in the interior of the cone and are glued to the cone, when in the off phase.

11. Space-like paths in the Minkowski spacetime

The criteria in Eq. (39) ensure that

$$\dot{q}_0^a [\mathbf{Q}_a^t \dot{\mathbf{Q}}_a - Q_a^0 Q_a^{0\prime} \dot{q}_0^a] = 0 \tag{89}$$

no matter whether in the on or in the off phase. Substituting Eqs. (11)–(15) into the above equation and using Eq. (23), we obtain

$$\beta \dot{\mathbf{q}}^t \dot{\mathbf{q}}^p = (\dot{\mathbf{q}}^p)^t \dot{\mathbf{q}}^p \geqslant 0. \tag{90}$$

Materials which satisfy such an inequality $\dot{\mathbf{q}}^t \dot{\mathbf{q}}^p \geqslant 0$ are said to be kinematically stable, e.g., Lubliner (1984). Using Eq. (23) and the above equation we obtain

$$\dot{q}_0^a = \sqrt{\beta \dot{\mathbf{q}}^t \dot{\mathbf{q}}^p}. \tag{91}$$

Squaring of the above equation and using Eqs. (17) and (14), we get

$$(\dot{q}_0^a)^2 = \dot{q}_0^a \frac{\beta \mathbf{Q}_a^t \dot{\mathbf{q}}}{Q_a^0}, \tag{92}$$

by which we have

$$(\dot{q}_0^a)^2 \leqslant \frac{\beta \dot{q}_0^a}{Q_a^0} \|\mathbf{Q}_a\| \|\dot{\mathbf{q}}\|, \tag{93}$$

and using Eq. (18) the bound of $\dot{q}_0^a$ can be derived,

$$\dot{q}_0^a \leqslant \beta \|\dot{\mathbf{q}}\| \tag{94}$$

no matter in the on or off phase. This inequality tells us that the maximum value of the specific dissipation power $\dot{\Lambda} = Q_{\mathrm{a}}^0 \dot{q}_0^a$ an admissible path in the state space may discharge is $\beta Q_{\mathrm{a}}^0 \|\dot{\mathbf{q}}\|$. (On the other hand, postulation (17) tells us that the minimum value of the specific dissipation power an admissible path may discharge is zero.)

The solution of Eq. (79) may be viewed as a path in $\mathbb{M}^{n+1}$, which traces along the null cone $\mathbf{X}_{\mathrm{a}}^{\mathrm{t}}\mathbf{g}\mathbf{X}_{\mathrm{a}} = 0$ given by Eq. (85). From which the derivative gives

$$\mathbf{X}_{\mathrm{a}}^{\mathrm{t}}\mathbf{g}\,\mathrm{d}\mathbf{X}_{\mathrm{a}} = 0. \tag{95}$$

Thus $\mathbf{X}_{\mathrm{a}}$ is M-orthogonal to $\mathrm{d}\mathbf{X}_{\mathrm{a}}$. Furthermore, upon using this equation and $\mathbf{X}_{\mathrm{a}}^{\mathrm{t}}\mathbf{g}\mathbf{X}_{\mathrm{a}} = 0$ we have

$$\sum_{i=1}^{n}(\mathrm{d}X_{\mathrm{a}}^i)^2 - (\mathrm{d}X_{\mathrm{a}}^0)^2 = \frac{1}{(X_{\mathrm{a}}^0)^2}\left[\sum_{i=1}^{n}(X_{\mathrm{a}}^i)^2\sum_{i=1}^{n}(\mathrm{d}X_{\mathrm{a}}^i)^2 - \left(\sum_{i=1}^{n}X_{\mathrm{a}}^i\,\mathrm{d}X_{\mathrm{a}}^i\right)^2\right]. \tag{96}$$

So that by utilizing the Schwartz inequality we obtain

$$(\mathrm{d}X_{\mathrm{a}})^2 := (\mathrm{d}\mathbf{X}_{\mathrm{a}})^{\mathrm{t}}\mathbf{g}\,\mathrm{d}\mathbf{X}_{\mathrm{a}} \geqslant 0. \tag{97}$$

Recalling that a path such that $(\mathrm{d}\mathbf{X}_{\mathrm{a}})^{\mathrm{t}}\mathbf{g}\,\mathrm{d}\mathbf{X}_{\mathrm{a}} > 0$ (resp. $= 0, < 0$) is called a space-like (resp. null, time-like) path in $\mathbb{M}^{n+1}$, we thereby conclude that the path $\mathbf{X}_{\mathrm{a}}(\tau')$, $\tau_{\mathrm{i}} < \tau' \leqslant \tau$, in the augmented state space is a space-like or null path in the Minkowski spacetime $\mathbb{M}^{n+1}$ no matter in the on or in the off phase. Indeed, Eq. (97) conveys a message that the nature of mixed-hardening elastoplasticity rejects time-like paths, so that the time-like metric convention has to be rejected to avoid an unreasonable negative squared length. This is the reason why we have adopted the space-like convention (84) for mixed-hardening elastoplasticity.

12. The proper orthochronous Lorentz group

In this section we concentrate on the on phase to bring out internal symmetry inherent in the model in the on phase. Denote by I_{on} an open, maximal, continuous proper time interval during which the mechanism of plasticity is on exclusively. The solution of the augmented state equation (79) can be expressed as in the following augmented state transition formula:

$$\mathbf{X}_{\mathrm{a}}(\tau) = [\mathbf{G}(\tau)\mathbf{G}^{-1}(\tau_1)]\mathbf{X}_{\mathrm{a}}(\tau_1), \quad \forall\ \tau, \tau_1 \in I_{on}, \tag{98}$$

in which $\mathbf{G}(\tau)$, called the fundamental matrix of Eq. (79), is a square matrix of order $n+1$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{G}(\tau) = \mathbf{B}(\tau)\mathbf{G}(\tau), \tag{99}$$

$$\mathbf{G}(0) = \mathbf{I}_{n+1}. \tag{100}$$

On the other hand, from Eqs. (80) and (84) it is easy to verify that the control matrix $\mathbf{B}$ in the on phase satisfies

$$\mathbf{B}^{\mathrm{t}}\mathbf{g} + \mathbf{g}\mathbf{B} = \mathbf{0}. \tag{101}$$

By Eqs. (101) and (99) we find

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[\mathbf{G}^{\mathrm{t}}(\tau)\mathbf{g}\mathbf{G}(\tau)] = \mathbf{0}.$$

From Eq. (100) we have $\mathbf{G}^{\mathrm{t}}(\tau)\mathbf{g}\mathbf{G}(\tau) = \mathbf{I}_{n+1}^{\mathrm{t}}\mathbf{g}\mathbf{I}_{n+1} = \mathbf{g}$ at $\tau = 0$, and thus prove that

$$\mathbf{G}^{\mathrm{t}}(\tau)\mathbf{g}\mathbf{G}(\tau) = \mathbf{g} \tag{102}$$

for all $\tau \in I_{\mathrm{o}n}$. Take determinants of both sides of the above equation, getting

$$(\det\ \mathbf{G})^2 = 1, \tag{103}$$

so that $\mathbf{G}$ is invertible. The 00 entry of the matrix equation (102) is $\sum_{i=1}^{n}(G_0^i)^2 - (G_0^0)^2 = -1$, from which

$$(G_0^0)^2 \geqslant 1. \tag{104}$$

Here $G_j^i, i, j = 1, \ldots, n, 0$, is the mixed ij-entry of the matrix $\mathbf{G}$. Since $\det\ \mathbf{G} = -1$ or $G_0^0 \leqslant -1$ would violate Eq. (100), it turns out that

$$\det\ \mathbf{G} = 1, \tag{105}$$

$$G_0^0 \geqslant 1. \tag{106}$$

In summary, $\mathbf{G}$ has the three characteristic properties explicitly expressed by Eqs. (102), (105) and (106).

Recall that the complete homogeneous Lorentz group $\mathrm{O}(n,1)$ is the group of all invertible linear transformations in Minkowski spacetime which leave the Minkowski metric invariant, and that the proper orthochronous Lorentz group $\mathrm{SO}_{\mathrm{o}}(n,1)$ is a subgroup of $\mathrm{O}(n,1)$ in which the transformations are proper (i.e., orientation preserving, namely the determinants of the transformations being $+1$) and orthochronous (i.e., time-orientation preserving, namely the 00 entry of the matrix representations of the transformations being positive) (see, e.g., Cornwell, 1984). Hence, in view of the three characteristic properties we conclude that the fundamental matrix $\mathbf{G}$ belongs to the proper orthochronous Lorentz group $\mathrm{SO}_{\mathrm{o}}(n,1)$. So the matrix function $\mathbf{G}(\tau)$ of proper time $\tau \in I_{\mathrm{o}n}$ may be viewed as a connected path of the Lorentz group. Furthermore, by Eq. (101), $\mathbf{B}$ is an element of the real Lie algebra $so(n,1)$ of the Lorentz group $\mathrm{SO}_{\mathrm{o}}(n,1)$.

From Eq. $(78)_1$, $\mathrm{d}X_{\mathrm{a}}^0/\mathrm{d}\tau > 0$ strictly when the mechanism of plasticity is on; hence, [3]

$$X_{\mathrm{a}}^0(\tau) > X_{\mathrm{a}}^0(\tau_1) > X_{\mathrm{a}}^0(\tau_0) = Q_{\mathrm{a}}^0, \qquad \forall\tau > \tau_1 > \tau_0, \quad \tau, \tau_1 \in I_{\mathrm{o}n}, \tag{107}$$

which means that in the sense of irreversibility there exists future-pointing proper time-orientation from the augmented states $\mathbf{X}_{\mathrm{a}}(\tau_1)$ to $\mathbf{X}_{\mathrm{a}}(\tau)$. Moreover, such time-orientation is a causal one, because the augmented state transition formula (98) and inequality (107) establish a *causality relation* between the two augmented states $\mathbf{X}_{\mathrm{a}}(\tau_1)$ and $\mathbf{X}_{\mathrm{a}}(\tau)$ in the sense that the preceding augmented state $\mathbf{X}_{\mathrm{a}}(\tau_1)$ influences the following augmented state $\mathbf{X}_{\mathrm{a}}(\tau)$ according to formula (98). Accordingly, the augmented state $\mathbf{X}_{\mathrm{a}}(\tau_1)$ chronologically and causally precedes the augmented state $\mathbf{X}_{\mathrm{a}}(\tau)$. This is indeed a common property for all models with inherent symmetry of the proper orthochronous Lorentz group. By this symmetry a core connection among irreversibility, the time arrow of evolution, and causality has thus been established for plasticity in the on phase.

13. The Poincaré group

The proper orthochronous Lorentz group $\mathrm{SO}_{\mathrm{o}}(n,1)$ constructed in the above is acting on the space $\mathbf{X}_{\mathrm{a}}$; and what is the effect of the kinematic hardening on the group structure? In order to reply this question we

[3] From Eqs. (17), (26) and (73) it follows that $X_{\mathrm{a}}^0(\tau) \geqslant X_{\mathrm{a}}^0(\tau') \geqslant X_{\mathrm{a}}^0(\tau_{\mathrm{i}}) \geqslant X_{\mathrm{a}}^0(\tau_0) = Q_{\mathrm{a}}^0$ for all $\tau \geqslant \tau' \geqslant \tau_{\mathrm{i}} \geqslant \tau_0$, applicable to both the on and off phases. Recall that τ_0 is the zero-value proper time at which all relevant values including $q_0^a(\tau_0) = 0$.

return to the derivation of Eq. (31) as follows. First, from Eqs. (15), (14) and (27), the generalized back stress is integrated as follows:

$$\mathbf{Q}_\mathrm{b}(t) = \mathbf{Q}_\mathrm{b}(t_\mathrm{i}) + \frac{G_\mathrm{b}(q_0^a(t), q_0^a(t_\mathrm{i}))}{k_\mathrm{e}(q_0^a(t_\mathrm{i}))}\mathbf{Q}_\mathrm{a}(t_\mathrm{i}) + \int_{t_i}^{t} G_\mathrm{b}(q_0^a(t), q_0^a(\xi))\dot{\mathbf{q}}(\xi)\,\mathrm{d}\xi, \tag{108}$$

where

$$C_\mathrm{p}(q_0^a) := \int_0^{q_0^a} \frac{k_\mathrm{p}(p)}{Y(p)Q_\mathrm{a}^0(p)}\,\mathrm{d}p, \tag{109}$$

$$G_\mathrm{b}(p_1, p_2) := k_\mathrm{e}(p_2)Y(p_2)\left[C_\mathrm{p}(p_1) - C_\mathrm{p}(p_2)\right]. \tag{110}$$

Both C_p and G_b are material functions. In terms of

$$G_\mathrm{a}(p_1, p_2) := \frac{k_\mathrm{e}(p_2)Y(p_2)}{Y(p_1)}, \tag{111}$$

it can be verified from Eqs. (26), (32), (33), (110) and (109) that

$$G^\mathrm{s}(p_1, p_2) := k_\mathrm{e}(p_2)\{1 - Y(p_2)[C_\mathrm{e}(p_1) - C_\mathrm{e}(p_2)]\} = G_\mathrm{a}(p_1, p_2) + G_\mathrm{b}(p_1, p_2). \tag{112}$$

Combining Eqs. (27) and (108) and noting Eqs. (12) and (112) yield Eq. (31) again. Equation (108) is an integral representation expressing the generalized back stress in terms of the generalized strain rate history.

Now, introduce the space $\mathbf{X}$ with the following vector decomposition:

$$\mathbf{X} = \mathbf{X}_\mathrm{a} + \mathbf{X}_\mathrm{b} := \begin{bmatrix} Y\mathbf{Q}_\mathrm{a} \\ YQ_\mathrm{a}^0 \end{bmatrix} + \begin{bmatrix} \mathbf{Q}_\mathrm{b} \\ 0 \end{bmatrix}. \tag{113}$$

The group acting on the space $\mathbf{X}$ is the semi-direct product of the translation $\mathbf{T}_{n+1}$ and the proper orthochronous Lorentz group $\mathrm{SO_o}(n, 1)$; usually such group is named the Poincaré group $\mathrm{ISO_o}(n, 1)$, or inhomogeneous Lorentz group (see, e.g., Kim and Noz, 1986).

14. Conformal spacetime

From Eqs. (11)–(13), (15) and (91) we have

$$(\mathrm{d}Q_\mathrm{a})^2 := \|\mathrm{d}\mathbf{Q}_\mathrm{a}\|^2 = k_\mathrm{e}^2\|\mathrm{d}\mathbf{q}\|^2 - (k_\mathrm{e} + k_\mathrm{p})[k_\mathrm{e} + k_\mathrm{p} + 2Q_\mathrm{a}^{0'}](\mathrm{d}q_0^a)^2, \tag{114}$$

which, with the aid of Eqs. (71)–(73), (39), (84) and (97), leads to

$$(\mathrm{d}X_\mathrm{a})^2 = Y^2[(\mathrm{d}Q_\mathrm{a})^2 - (Q_\mathrm{a}^{0'})^2(\mathrm{d}q_0^a)^2] \tag{115}$$

no matter in the on or in the off phase. Especially, in the on phase we have $\mathbf{Q}_\mathrm{a} \cdot \mathrm{d}\mathbf{Q}_\mathrm{a} = Q_\mathrm{a}^0 Q_\mathrm{a}^{0'}\mathrm{d}q_0^a$ by the plastic consistency condition, and for the nonperfect case, i.e., $Q_\mathrm{a}^{0'} \neq 0$, we have

$$(\mathrm{d}X_\mathrm{a})^2 = Y^2\left[\|\mathrm{d}\mathbf{Q}_\mathrm{a}\|^2 - \frac{(\mathbf{Q}_\mathrm{a} \cdot \mathrm{d}\mathbf{Q}_\mathrm{a})^2}{(Q_\mathrm{a}^0)^2}\right] = Y^2(Q_\mathrm{a}^{0'})^2\left[\frac{1}{(Q_\mathrm{a}^{0'})^2}(\mathrm{d}Q_\mathrm{a})^2 - (\mathrm{d}q_0^a)^2\right]. \tag{116}$$

The metric line element $(\mathrm{d}Q_\mathrm{a})^2/(Q_\mathrm{a}^{0'})^2 - (\mathrm{d}q_0^a)^2$ defined in the space $(\mathbf{Q}_\mathrm{a}^\mathrm{t}, q_0^a)$ indicates that the spatial sections expand (or contract) uniformly as described by the scalar function $Q_\mathrm{a}^{0'}(q_0^a)$. This form of the metric line element is manifestly conformal to the Minkowski space $\mathbf{X}_\mathrm{a}$ with a conformal scalar factor $Y(q_0^a)$. The space $(\mathbf{Q}_\mathrm{a}^\mathrm{t}, q_0^a)$ is known as a Robertson–Walker spacetime (see, e.g., Hawking and Ellis, 1973 and Birrell and Davies, 1982).

In this occasion we would like to point out that the metric considered by De Saxce and Hung (1985), who attempted to study plasticity models from a differential geometry view, is not a suitable one. Really, by identifying $(Q_{\mathrm{a}}^0)^2 = 2\bar{\sigma}^2/3$, $\mathrm{d}\Lambda = Q_{\mathrm{a}}^0 \mathrm{d}q_0^a = \mathrm{d}\kappa^0$ and $\mathrm{d}q_\alpha^{\mathrm{p}} = \mathrm{d}\kappa^\alpha$ with the notations used by De Saxce and Hung, we readily obtain

$$\mathrm{d}\sigma^2 = (\mathrm{d}\kappa^0)^2 - \frac{2\bar{\sigma}^2}{3}\delta_{\alpha\beta}\,\mathrm{d}\kappa^\alpha\,\mathrm{d}\kappa^\beta$$

from the flow rule (14). See also Eq. (5.2) proposed by De Saxce and Hung, *loc.cit.* The above metric vanishes no matter whether in the on or in the off phase. In this sense such metric is not a suitable measure for its full degeneracy. More precisely, it is not a metric and also not a hyperbolic metric. So, De Saxce and Hung asserting it as a hyperbolic metric on the base manifold is incorrect. Indeed, $\mathrm{d}\sigma^2 = 0$ is at most an identity derived from the associated flow rule.

Eq. (75) is the governing equation of $\mathbf{X}_{\mathrm{a}} = (Y\mathbf{Q}_{\mathrm{a}}^{\mathrm{t}}, YQ_{\mathrm{a}}^0)^{\mathrm{t}}$; and from Eqs. (71), (72), (26), (77) and (39) the governing equation of $(\mathbf{Q}_{\mathrm{a}}, Q_{\mathrm{a}}^0)$ is read as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\mathbf{Q}_{\mathrm{a}}\\ Q_{\mathrm{a}}^0\end{bmatrix} = \frac{k_{\mathrm{e}}}{Q_{\mathrm{a}}^0}\begin{bmatrix}\mathbf{0}_{n\times n} & \dot{\mathbf{q}}\\ \dot{\mathbf{q}}^{\mathrm{t}} & 0\end{bmatrix}\begin{bmatrix}\mathbf{Q}_{\mathrm{a}}\\ Q_{\mathrm{a}}^0\end{bmatrix} - \frac{\beta(k_{\mathrm{e}}+k_{\mathrm{p}})}{(Q_{\mathrm{a}}^0)^2}\mathbf{Q}_{\mathrm{a}}^{\mathrm{t}}\dot{\mathbf{q}}\begin{bmatrix}\mathbf{Q}_{\mathrm{a}}\\ Q_{\mathrm{a}}^0\end{bmatrix}. \tag{117}$$

However, $(\mathbf{Q}_{\mathrm{a}}^{\mathrm{t}}, Q_{\mathrm{a}}^0)^{\mathrm{t}}$ is not a suitable spacetime structure, since $\mathrm{d}Q_{\mathrm{a}}^0$ may be negative in the softening range of modeled material.

In order to obtain a suitable conformal spacetime structure, we introduce a scaling factor function $\rho(q_0^a)$ and consider the following augmented vector:

$$\mathbf{X}_{\mathrm{c}} = \begin{bmatrix}\mathbf{X}_{\mathrm{c}}^{\mathrm{s}}\\ X_{\mathrm{c}}^0\end{bmatrix} := \begin{bmatrix}\rho\mathbf{Q}_{\mathrm{a}}\\ \rho Q_{\mathrm{a}}^0\end{bmatrix}, \tag{118}$$

in which we need $\mathrm{d}(\rho Q_{\mathrm{a}}^0) > 0$ in the on phase, and simultaneously the cone is preserved, i.e.,

$$(\mathbf{X}_{\mathrm{c}})^{\mathrm{t}}\mathbf{g}\mathbf{X}_{\mathrm{c}} = 0. \tag{119}$$

Taking the time derivative of Eq. (118) and using (117) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\rho\mathbf{Q}_{\mathrm{a}}\\ \rho Q_{\mathrm{a}}^0\end{bmatrix} = \frac{\dot{\rho}}{\rho}\begin{bmatrix}\rho\mathbf{Q}_{\mathrm{a}}\\ \rho Q_{\mathrm{a}}^0\end{bmatrix} + \frac{k_{\mathrm{e}}}{Q_{\mathrm{a}}^0}\begin{bmatrix}\mathbf{0}_{n\times n} & \dot{\mathbf{q}}\\ \dot{\mathbf{q}}^{\mathrm{t}} & 0\end{bmatrix}\begin{bmatrix}\rho\mathbf{Q}_{\mathrm{a}}\\ \rho Q_{\mathrm{a}}^0\end{bmatrix} - \frac{\beta(k_{\mathrm{e}}+k_{\mathrm{p}})}{\rho(Q_{\mathrm{a}}^0)^2}\rho\mathbf{Q}_{\mathrm{a}}^{\mathrm{t}}\dot{\mathbf{q}}\begin{bmatrix}\rho\mathbf{Q}_{\mathrm{a}}\\ \rho Q_{\mathrm{a}}^0\end{bmatrix}. \tag{120}$$

If ρ satisfies

$$\frac{\dot{\rho}}{\rho} = \frac{k_{\mathrm{e}}}{Q_{\mathrm{a}}^0}\left[\frac{\beta(k_{\mathrm{e}}+k_{\mathrm{p}})}{\rho k_{\mathrm{e}} Q_{\mathrm{a}}^0} - 2\rho_0\right]\rho\mathbf{Q}_{\mathrm{a}}^{\mathrm{t}}\dot{\mathbf{q}}, \tag{121}$$

where ρ_0 is a constant whose range to be determined below, then in the space $\mathbf{X}_{\mathrm{c}}$ the governing equations are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\rho\mathbf{Q}_{\mathrm{a}}\\ \rho Q_{\mathrm{a}}^0\end{bmatrix} = \frac{k_{\mathrm{e}}}{Q_{\mathrm{a}}^0}\begin{bmatrix}\mathbf{0}_{n\times n} & \dot{\mathbf{q}}\\ \dot{\mathbf{q}}^{\mathrm{t}} & 0\end{bmatrix}\begin{bmatrix}\rho\mathbf{Q}_{\mathrm{a}}\\ \rho Q_{\mathrm{a}}^0\end{bmatrix} - \frac{2k_{\mathrm{e}}\rho_0}{Q_{\mathrm{a}}^0}\rho\mathbf{Q}_{\mathrm{a}}^{\mathrm{t}}\dot{\mathbf{q}}\begin{bmatrix}\rho\mathbf{Q}_{\mathrm{a}}\\ \rho Q_{\mathrm{a}}^0\end{bmatrix}. \tag{122}$$

By using the proper time τ defined in Eq. (77) the above equations are further refined to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix}\rho\mathbf{Q}_{\mathrm{a}}\\ \rho Q_{\mathrm{a}}^0\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{n\times n} & \dot{\mathbf{q}}\\ \dot{\mathbf{q}}^{\mathrm{t}} & 0\end{bmatrix}\begin{bmatrix}\rho\mathbf{Q}_{\mathrm{a}}\\ \rho Q_{\mathrm{a}}^0\end{bmatrix} - 2\rho_0\rho\mathbf{Q}_{\mathrm{a}}^{\mathrm{t}}\dot{\mathbf{q}}\begin{bmatrix}\rho\mathbf{Q}_{\mathrm{a}}\\ \rho Q_{\mathrm{a}}^0\end{bmatrix}. \tag{123}$$

The above differential equations system is one of the conformal type, e.g., Anderson et al. (1982), with the $(n+1)$-dimensional conformal parameters $(\rho_0\dot{\mathbf{q}}, 0)$. The group generated from the above differential equations system is a special type of conformal group.

Eq. (121), after substituting $\dot{\rho} = \rho' \dot{q}_0^a$ and Eq. $(39)_1$ for $\mathbf{Q}_a^t \dot{\mathbf{q}} = Q_a^0 \dot{q}_0^a / \beta$ and cancelling $\dot{q}_0^a$'s on both sides, can be changed to the following Riccati differential equation:

$$\rho' = \frac{k_e + k_p}{Q_a^0} \rho - \frac{2k_e \rho_0}{\beta} \rho^2. \tag{124}$$

Using Eq. (26) the solution of ρ is found to be

$$\rho(q_0^a) = \frac{Y(q_0^a)}{1 + 2\rho_0 \int_0^{q_0^a} \frac{k_e(p)Y(p)}{\beta(p)} \, dp}. \tag{125}$$

On the other hand, from Eqs. (61) and (42) we have

$$\frac{d(YQ_a^0)}{dt} = \frac{k_e}{Q_a^0} \dot{Z} = \frac{k_e Y}{\beta} \dot{q}_0^a > 0 \tag{126}$$

in the on phase, such that the integral in Eq. (125) can be obtained explicitly,

$$\rho(q_0^a) = \frac{Y(q_0^a)}{1 - 2\rho_0 Q_a^0(0) + 2\rho_0 Y(q_0^a) Q_a^0(q_0^a)}. \tag{127}$$

When $\rho_0 = 0$ we have $\rho = Y$, and the differential equations system (79) is recovered from Eq. (123). We also note that ρ_0 can not be a negative constant value; otherwise, the above ρ will blow-up when Y increases to a certain positive value.

From Eq. (127) the relation of $\mathbf{X}_a$ and $\mathbf{X}_c$ through Eqs. (73) and (118) is available,

$$\mathbf{X}_c = \frac{1}{1 - 2\rho_0 Q_a^0(0) + 2\rho_0 X_a^0} \mathbf{X}_a. \tag{128}$$

Furthermore, the metrics in both the spaces $\mathbf{X}_c$ and $\mathbf{X}_a$ can be proved to be related conformally as follows:

$$(dX_c)^2 = \frac{[1 - 2\rho_0 X_c^0]^2}{[1 - 2\rho_0 Q_a^0(0)]^2} (dX_a)^2 = \frac{1}{[1 - 2\rho_0 Q_a^0(0) + 2\rho_0 X_a^0]^2} (dX_a)^2, \tag{129}$$

where

$$(dX_c)^2 := (d\mathbf{X}_c)^t \mathbf{g} \, d\mathbf{X}_c. \tag{130}$$

Finally, from Eq. (128) it is easy to prove that

$$\dot{X}_c^0 = \frac{[1 - 2\rho_0 Q_a^0(0)]}{[1 - 2\rho_0 Q_a^0(0) + 2\rho_0 X_a^0]^2} \dot{X}_a^0 = \frac{[1 - 2\rho_0 X_c^0]^2}{1 - 2\rho_0 Q_a^0(0)} \dot{X}_a^0. \tag{131}$$

For the purpose of $X_c^0 > 0$ to be an irreversible parameter as well, i.e., $\dot{X}_c^0 > 0$, by Eqs. (126) and (73) we require $1 - 2\rho_0 Q_a^0(0) > 0$, that is,

$$0 \leqslant \rho_0 < \frac{1}{2Q_a^0(0)}. \tag{132}$$

Thus, under this condition we have introduced another irreversible parameter X_c^0 as the temporal component of the conformal spacetime, the relations of which to the other irreversible parameters are listed in Table 1. It includes two dimensionless irreversible parameters q_0^a and Y, and also four stress dimension irreversible parameters Z, Λ, X_a^0 and X_c^0.

The hyperbolic geometric models that we studied for the mixed-hardening elastoplasticity are governed by the conformally isormorphic spacetimes. They are all differentiable manifold endowed with a pseudo-Riemann metric, which assigns at each spacetime point $\mathbf{X}_c$ an indefinite symmetric inner product on the tangent space, which varying differentiably with $\mathbf{X}_c$.

15. Conclusions

In this paper we have established two types formulations for mixed-hardening elastoplastic model of the generalized stress and generalized strain: the flow model characterized by Eqs. (11)–(18) and the two-phase linear differential system characterized by Eqs. (79) and (80). Even though the governing equations of the generalized active stress are nonlinear in the n-dimensional space $\mathbf{Q}_a$ we have found that it can be transformed to linear differential equations in the augmented $(n+1)$-dimensional space $\mathbf{X}_a$. In this space not only the nonlinearity of the model can be unfolded, but also an intrinsic spacetime structure of the Minkowskian type can be revealed, merely replacing the inequality in space $\mathbf{Q}_a$ to the inequality in the augmented space $\mathbf{X}_a$. We also pointed out that the state matrix $\mathbf{B}$ is an element of the Lie algebra of the proper orthochronous Lorentz group, hence the state transition matrix generated from the linear differential equations was proved to be a type of the proper orthochronous Lorentz group. In the frame of the Minkowski spacetime we have further proved that the action of the kinematic rule in the mixed-hardening model causes a translation of the state $\mathbf{X}$, which amounts to extend the proper orthochronous Lorentz group to the proper orthochronous Poincaré group.

We have introduced a conformal factor ρ in Eq. (127) and thus a conformal spacetime $\mathbf{X}_c$ is derived. The analytic models of hyperbolic geometry that we studied for the mixed-hardening elastoplasticity are governed by the conformally isormorphic spacetimes. They are all differentiable manifold endowed with a pseudo-Riemann metric, which assigns at each spacetime point an indefinite symmetric inner product on the tangent space, which varying differentiably. Mathematically speaking, the pseudo-Riemannian manifold is a suitable underlying spacetime model for the mixed-hardening elastoplasticity.

Even we dealt only with mixed-hardening effect on the elastoplastic model without considering a more inclusive Armstrong-Frederick kinematic hardening rule, [4] and/or large deformation, etc.; however, we may further consider more sophisticated group actions on pseudo-Riemannian manifolds and thus makes an explicit use of the powerful group-theoretic method to study plasticity from a global view. In this regard, the present paper may open the way to plasticity research in a new direction.

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[4] Armstrong and Frederick (1966) have proposed a nonlinear kinematic hardening rule $\dot{\mathbf{Q}}_b = k_p\dot{\mathbf{q}}^p - k_b\dot{q}_0^a\mathbf{Q}_b$ with $k_b > 0$ to account of the saturation of $\mathbf{Q}_b$ under proportional loading conditions.

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